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Pion propagation in real time field theory at finite temperature

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Abstract. We describe how the thermal counterpart of a vacuum two-point function may be obtained in the real time formalism in a simple way by using directly the 2×2 matrices that different elements acquire in this formalism. Using this procedure we calculate the analytic (single component) thermal amplitude for the pion pole term in the ensemble average of two axial-vector currents to two loops in chiral perturbation theory. The general expressions obtained for the effective mass and the decay constants of the pion are evaluated in the chiral and the non-relativistic limits.

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1 Introduction

The real time thermal field theory is apparently complicated by the fact that all two-point functions in this formalism assume the form of 2×2 matrices [1–3]. These matrices, however, have simple structures: if we factor out certain matrices depending on the distribution function only, these become diagonal, each with essentially a single independent element with proper analytic properties. But in actual computations one tends to ignore the matrix structure, starting instead with the so-called physical 11-element, encountering though summation over indices at all interaction vertices in a Feynman graph. Such a procedure leads to pieces of ill-defined products of components of the matrix propagator that must be combined to get a well-defined quantity. Further the 11-component does not have a simple analytic structure.

In this work we show that it is both simple and elegant to work with the matrix amplitudes. All one has to do is to write out the usual vacuum amplitude. The thermal matrix amplitude is then obtained by replacing its elements like the propagator, the self-energy and the vertices, by the corresponding matrices. Factorizing these matrices as mentioned above, we immediately get the analytic amplitude representing the dynamics of the system in the heat bath.

Here we apply this procedure to calculate the pion pole term in the two-point function of the axial-vector currents to two loops in chiral perturbation theory [4, 5]. This problem was studied earlier by several authors [6–10], in particular, by Toublan [11]. After obtaining the analytic amplitude, we follow him to find the pion pole position and the residue. We then find these pole parameters in the chiral limit, in agreement with his results. We also evaluate them in the non-relativistic region.

In Sect. 2 we write down the effective chiral Lagrangian to fourth order, needed to obtain all the required vertices. In the next section, Sect. 3, we obtain the vacuum amplitude from all the Feynman graphs up to two loops contributing to the pion pole term of the two-point function. The corresponding thermal amplitude is obtained in Sect. 4, from which we derive the effective parameters, namely the pion mass and the decay constants at finite temperature. These expressions are evaluated analytically in Sect. 5 in the high and low temperature limits. Finally we bring out the main features of our work in Sect. 6.

Appendix A constitutes an essential part of this work. Reviewing briefly the real time thermal field theory, we discuss here at length how the vacuum amplitude for an individual Feynman graph may be converted into its thermal counterpart. In Appendix B we write the integrals appearing in the non-factorizable amplitudes. In the last appendix, Appendix C, we collect the results for the relevant integrals in the high and the low temperature region.

2 Chiral perturbation theory

We consider the QCD Lagrangian for the doublet of light quarks, u and d. In the absence of their masses, it has chiral symmetry, being invariant under $SU(2)_R \times SU(2)_L$. This symmetry is supposed to be broken spontaneously to $SU(2)_V$ of ordinary isospin, generating the massless pions as the Goldstone bosons.

In the physical case of non-zero quark masses, chiral symmetry is also broken explicitly to the same isospin sub-

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group, if we neglect the mass difference between u and d quarks. The pions become the pseudo-Goldstone bosons acquiring mass M, given by $M^2 = 2m_q B$ to lowest order, where B is related to the quark condensate in vacuum, whose dynamical generation leads to the spontaneous symmetry breaking.

As already stated, we are interested in the pion pole term in the two-point function of the axial-vector currents,

$$A_\mu^a = \bar{q}\gamma_\mu\gamma_5\frac{\tau^a}{2}q\,, \quad a = 1,2,3\,,$$

evaluated in chiral perturbation theory, the effective theory of QCD at low energy. Here τ^a are the Pauli matrices. Such functions are best calculated in the external field method, in which one introduces in the original QCD Lagrangian an external field $a_a^\mu(x)$ coupled to $A_\mu^a(x)$ as well as a field $v_a^\mu(x)$ for the vector currents [4, 5]. The global chiral symmetry is then promoted to a local one, assuming appropriate transformation properties of the external fields.

In the effective theory, the pion fields may be collected in the form of an unitary matrix,

$$U(x) = \mathrm{e}^{\mathrm{i}\varphi^a(x)\tau^a/F}.$$

The constant F may be identified as the pion decay constant in the chiral limit. The local symmetry requires us to replace the ordinary derivative by the covariant one,

$$D_\mu U = \partial_\mu U - \mathrm{i}\{a_\mu, U\}\,, \tag{1}$$

where, for our purpose, we retain only the external field $a_\mu(x)$. The two-point function of $A_\mu(x)$ is now obtained as the coefficient of the quadratic term in $a_\mu(x)$ in the perturbative evaluation of the generating functional with the effective Lagrangian.

As a non-renormalizable theory, the effective Lagrangian consists of a series of terms with an increasing number of derivatives and/or quark mass factors,

$$\mathcal{L}_{\text{eff}} = \mathcal{L}^{(2)} + \mathcal{L}^{(4)} + \ldots\,.$$

The leading term is given by

$$\mathcal{L}^{(2)} = \frac{F^2}{4}\left\{\langle D_\mu U^\dagger D^\mu U\rangle + M^2\langle U + U^\dagger\rangle\right\}, \tag{2}$$

where $\langle A\rangle$ denotes the trace of the 2×2 matrix A. Here the first term is invariant under the chiral transformations. The second term represents explicit symmetry breaking due to the quark mass term in the QCD Lagrangian.

The next, non-leading piece in $\mathcal{L}_{\text{eff}}$ is [4, 5, 12]

$$\begin{aligned}\mathcal{L}^{(4)} = &\frac{1}{4}l_1\langle D_\mu U^\dagger D^\mu U\rangle^2 + \frac{1}{4}l_2\langle D_\mu U^\dagger D_\nu U\rangle\langle D^\mu U^\dagger D^\nu U\rangle \\ &+ \frac{1}{8}l_4 M^2\langle D_\mu U^\dagger D^\mu U\rangle\langle U + U^\dagger\rangle \\ &+ \frac{1}{16}(l_3 + l_4)M^4\langle U + U^\dagger\rangle^2.\end{aligned} \tag{3}$$

It provides counterterms necessary to renormalize the one-loop graphs with vertices from $\mathcal{L}^{(2)}$. Thus the bare coupling constants $l_1, \ldots, l_4$ contain a pole at $d = 4$ in dimensional regularization. The coefficients of these poles may be determined by evaluating all the one-loop graphs. Alternatively, these may be obtained directly by calculating the short distance behavior of the generating functional to one loop [4, 5]. Adopting the notation introduced by the authors of this reference, the renormalized coupling constants $\bar{l}_1, \cdots, \bar{l}_4$ are defined by

$$l_i = \gamma_i\left(\lambda + \frac{1}{32\pi^2}\bar{l}_i\right), \tag{4}$$

with

$$\gamma_1 = \frac{1}{3}\,, \quad \gamma_2 = \frac{2}{3}\,, \quad \gamma_3 = -\frac{1}{2}\,, \quad \gamma_4 = 2\,.$$

The pole is contained in λ,

$$\lambda = \frac{M^{d-4}}{(4\pi)^2}\left(\frac{1}{d-4} - \frac{1}{2}[\ln 4\pi + \Gamma'(1) + 1] + O(d-4)\right). \tag{5}$$

Up to the factor $\gamma_i/32\pi^2$, the constants $\bar{l}_i$ are running coupling constants at the scale M. The M-dependence of $\bar{l}_i$ can be made explicit by relating these to l_i^r, the renormalized coupling constants at any other scale μ,

$$l_i^r = \frac{\gamma_i}{32\pi^2}\left(\bar{l}_i + \ln\frac{M^2}{\mu^2}\right). \tag{6}$$

In Sect. 5 we shall use this equation to show the finiteness of the pion pole parameters in the chiral limit.

The renormalization of the two-loop graphs would require vertices from the next higher piece, $\mathcal{L}^{(6)}$ in $\mathcal{L}_{\text{eff}}$. But we do not need it, as we are not interested in the amplitude in vacuum to this order, but only in its temperature dependent part.

3 Vacuum amplitude

Here we obtain the pion pole contribution to the vacuum two-point function of the axial-vector current,

$$\delta^{ab}T_{\mu\nu}(q) = \mathrm{i}\int \mathrm{d}^4x e^{\mathrm{i}qx}\langle 0|TA_\mu^a(x)A_\nu^b(0)|0\rangle_{\pi\ \text{pole}}\,, \tag{7}$$

by calculating all the Feynman graphs up to two loops with the interaction vertices given by $\mathcal{L}^{(2)}$ and $\mathcal{L}^{(4)}$. These graphs are conveniently divided into four groups, as shown in Figs. 1–4.

Let us first derive the familiar one-loop results. The vertex correction graphs (b) and (c) of Fig. 1 modify the residue of the free amplitude of graph (a) to give

$$T_{\mu\nu}^{(1\mathrm{a,b,c})}(q) = q_\mu q_\nu F^2\{1 + 4\eta(3l_4 - 2J)/3\}\mathrm{i}\Delta(q)\,, \tag{8}$$

where $\eta = \frac{M^2}{F^2}$ is an expansion parameter and $\Delta(q)$ is the free pion propagator, $\Delta(q) = \mathrm{i}/(q^2 - M^2 + \mathrm{i}\varepsilon)$. J is a divergent one-loop integral,

$$J(M) = \frac{1}{M^2}\int\frac{\mathrm{d}^4k}{(2\pi)^4}\Delta(k) \equiv 2\lambda\,, \tag{9}$$

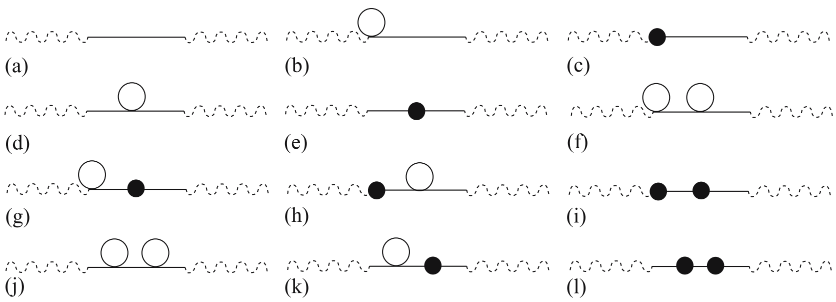

Fig. 1. The free amplitude with corrections from one-loop graphs along with counterterm graphs and those two-loop graphs that are iterations of the former ones. Vertices of $\mathcal{L}^{(2)}$ and $\mathcal{L}^{(4)}$ are shown as *points* and *filled circles* respectively. *Wavy* and *straight lines* denote axial current and pion respectively

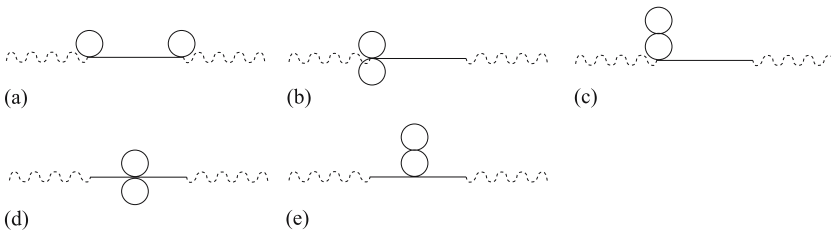

Fig. 2. Remaining factorizable two-loop graphs with vertices from $\mathcal{L}^{(2)}$ only

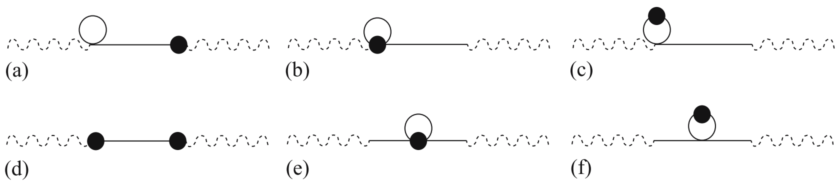

Fig. 3. Further counterterm graphs

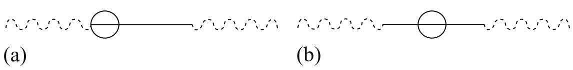

Fig. 4. Non-factorizable two-loop graphs

with λ given by (5). To include the self-energy graphs, it is convenient to introduce here the well-known Dyson–Schwinger equation for the complete propagator $\Delta'(q)$,

$$\Delta'(q) = \Delta(q) + \Delta(q)(-\mathrm{i}\Sigma(q))\Delta'(q), \tag{10}$$

where the self-energy part Σ of graphs (d) and (e) of Fig. 1 is given by

$$\Sigma(q) = -2\eta(3l_4 - J)(q^2 - M^2)/3 + F^2\eta^2(4l_3 + J)/2\,. \tag{11}$$

Equation (10) may be solved by iteration,

$$\begin{aligned}\Delta'(q) = \Delta(q) &+ \Delta(q)(-\mathrm{i}\Sigma(q))\Delta(q)\\ &+ \Delta(q)\{-\mathrm{i}\Sigma(q)\Delta(q)\}^2 + \ldots\end{aligned} \tag{12}$$

or in closed form,

$$\Delta'(q) = \frac{\Delta(q)}{1 + \mathrm{i}\Sigma(q)\Delta(q)}\,. \tag{13}$$

The self-energy correction is now included in (8) by replacing $\Delta(q)$ with $\Delta'(q)$. Thus we get the one-loop result for the pion pole,

$$T_{\mu\nu}(q) = -q_\mu q_\nu \frac{F_\pi^2}{q^2 - M_\pi^2 + \mathrm{i}\varepsilon}\,,$$

with

$$M_\pi^2 = M^2\{1 + 2\eta(l_3 + J/4)\} = M^2(1 - \eta\bar{l}_3/32\pi^2)\,, \tag{14}$$

$$F_\pi = F\{1 + \eta(l_4 - J)\} = F(1 + \eta\bar{l}_4/16\pi^2)\,, \tag{15}$$

on using (4).

We now include the two-loop graphs. First consider those of Figs. 1–3, that are actually products of two one-loop parts. The total contribution of all these graphs may be put in the form

$$\begin{aligned}T_{\mu\nu}^{(1+2+3)}(q) = q_\mu q_\nu F^2 \sum_{n=1}^{3} &\{\gamma_n \mathrm{i}\Delta(q) + \sigma_n M^2\Delta^2(q)\}\\ &+ 8F^2\eta^2(l_1 + 2l_2)(q_\mu J_{\nu\lambda} + q_\nu J_{\mu\lambda})q^\lambda \mathrm{i}\Delta(q)\,,\end{aligned} \tag{16}$$

where we show separately the sums of contributions of all graphs in each of Figs. 1–3. Thus the sum of graphs in Fig. 1 is given by the $n=1$ term[1]. Note here that the two-loop graphs from (f) to (l), being iterations of graphs from (b) to (e), are automatically included in (13). But we prefer to write them explicitly in (16), getting

$$\begin{aligned}\gamma_1 &= 1+2\eta(l_4-J)-4\eta^2(l_4-J)(3l_4-J)/3\,,\\ \sigma_1 &= \eta(4l_3+J)/2-2\eta^2 J(4l_3+J)/3\,,\end{aligned} \tag{17}$$

where we cancel factors as $(q^2-m^2)\Delta^2(q)=\mathrm{i}\Delta(q)$, which can also be justified at finite temperatures. In this way we get directly the shifts in the residue and the pole position from the graphs of Fig. 1 as $F^2(\gamma_1-1)$ and $M^2\sigma_1/\gamma_1$ respectively. Next, the graphs of Fig. 2 contain only the vertices of $\mathcal{L}^{(2)}$, and their sum is given by the $n=2$ term, with

$$\gamma_2=\eta^2 J(8J+3J')/3\,,\quad \sigma_2=-\eta^2 J(3J+2J')/8\,, \tag{18}$$

where we encounter a new divergent integral related to the earlier one:

$$J'(M)=\mathrm{i}\int\frac{\mathrm{d}^4k}{(2\pi)^4}\Delta^2(k)=-\frac{\partial}{\partial M^2}(M^2 J)=-2\lambda-\frac{1}{16\pi^2}\,. \tag{19}$$

Lastly the sum of graphs of Fig. 3 with vertices from $\mathcal{L}^{(2)}$ and $\mathcal{L}^{(4)}$ is given by the term $n=3$ with

$$\begin{aligned}\gamma_3 &= \eta^2\left\{(36l_1+12l_2-25l_4)J+12l_3J'+12l_4^2\right\}/3\,,\\ \sigma_3 &= -\eta^2\left\{(36l_1+12l_2+16l_3-3l_4)J+3l_3J'\right.\\ &\quad \left.+24(l_1+2l_2)q^\lambda q^\sigma J_{\lambda\sigma}/M^2\right\}/3\,,\end{aligned} \tag{20}$$

together with the remaining term in (16), where we have still another divergent integral,

$$J_{\mu\nu}(M)=\frac{1}{M^4}\int\frac{\mathrm{d}^4k}{(2\pi)^4}k_\mu k_\nu\Delta(k)\,. \tag{21}$$

Actually this term is also proportional to $q_\mu q_\nu$, once the integral is evaluated. But in view of its extension to finite temperature, we keep it as such.

Finally we have the amplitude from the non-factorizable two-loop graphs of Fig. 4,

$$\begin{aligned}T^{(4)}_{\mu\nu}(q) &= -\frac{2\mathrm{i}}{9F^2}\{q_\mu\Delta(q)\Gamma_\nu(q)+\Gamma_\mu(q)\Delta(q)q_\nu\}\\ &\quad+\frac{1}{18F^2}q_\mu q_\nu\Delta(q)\Sigma(q)\Delta(q)\,,\end{aligned} \tag{22}$$

where the vertex function $\Gamma_\mu(q)$ of graph (a) is

$$\begin{aligned}\Gamma_\mu(q) &= \mathrm{i}\int\frac{\mathrm{d}^4k_1}{(2\pi)^4}\frac{\mathrm{d}^4k_2}{(2\pi)^4}(2q_\mu-3k_{1\mu}-3k_{2\mu})f\\ &\quad\times\Delta(k_1)\Delta(k_2)\Delta(q-k_1-k_2)\,,\end{aligned} \tag{23}$$

[1] A piece, namely $-q_\mu q_\nu(F^2/4)\eta^2(J+4l_3)^2M^4\mathrm{i}\Delta^3(q)$, is omitted here, as it is automatically included when we put the $n=1$ term in the form of a simple pole.

and the self-energy function $\Sigma(q)$ of graph (b) is

$$\begin{aligned}\Sigma(q) &= -\mathrm{i}\int\frac{\mathrm{d}^4k_1}{(2\pi)^4}\frac{\mathrm{d}^4k_2}{(2\pi)^4}(3M^4+2f^2)\\ &\quad\times\Delta(k_1)\Delta(k_2)\Delta(q-k_1-k_2)\,,\end{aligned} \tag{24}$$

with f standing for the function

$$f(q,k_1,k_2)=k_1^2+k_2^2+4k_1k_2+M^2+2q(k_1+k_2)-2q^2.$$

The order of the factors in (22) is in anticipation of their matrix structures of the thermal amplitudes in the next section.

In [11] the vertex and self-energy integrals have been cast in a particularly convenient form using the symmetries of the integrands under the interchange of the integration variables. Thus if one defines

$$\begin{aligned}K(q) &= \frac{\mathrm{i}}{M^2}\int\frac{\mathrm{d}^4k_1}{(2\pi)^4}\frac{\mathrm{d}^4k_2}{(2\pi)^4}\\ &\quad\times\Delta(k_1)\Delta(k_2)\Delta(q-k_1-k_2)\,,\end{aligned} \tag{25}$$

$$\begin{aligned}K_{\mu\nu}(q) &= \frac{\mathrm{i}}{M^4}\int\frac{\mathrm{d}^4k_1}{(2\pi)^4}\frac{\mathrm{d}^4k_2}{(2\pi)^4}k_{1\mu}k_{1\nu}\\ &\quad\times\Delta(k_1)\Delta(k_2)\Delta(q-k_1-k_2)\,,\end{aligned} \tag{26}$$

they may be written as

$$\Gamma_\mu(q)=-2F^4\eta^2\left[q_\mu(3J^2-K)+9K_{\mu\varrho}q^\varrho\right]\,, \tag{27}$$

$$\begin{aligned}\Sigma(q) &= -F^4\eta^2\left[4(9J^2-4K)(q^2-M^2)\right.\\ &\quad\left.+3M^2(8J^2-K)+72q^\varrho q^\sigma K_{\varrho\sigma}\right]\,.\end{aligned} \tag{28}$$

Then (22) simplifies to

$$\begin{aligned}T^{(4)}_{\mu\nu} &= q_\mu q_\nu F^2\eta^2\left\{\frac{2}{3}J^2\mathrm{i}\Delta(q)\right.\\ &\quad\left.-\frac{1}{6}\left(8J^2-K+24q^\varrho q^\sigma K_{\varrho\sigma}/M^2\right)M^2\Delta^2(q)\right\}\\ &\quad+4F^2\eta^2(q_\mu K_{\nu\lambda}+q_\nu K_{\mu\lambda})q^\lambda\mathrm{i}\Delta(q)\,.\end{aligned} \tag{29}$$

The sum of the amplitudes (16) and (29), along with the one from the tree graphs with a single insertion of vertices from $\mathcal{L}^{(6)}$ (not calculated above), would give the complete, renormalized vacuum amplitude. One may then extend the one-loop results (14) and (15) for the pion pole parameters to two loops. Instead, however, we turn to the corresponding thermal amplitudes to find the temperature dependence of these parameters.

4 Thermal amplitude

The thermal (ensemble averaged) two-point function of the axial-vector current is a 2×2 matrix in the real time formalism, whose ij element, restricted to the pion pole, is

$$\begin{aligned}\left(T^{ab}_{\mu\nu}\right)_{ij} &= \mathrm{i}\int\mathrm{d}^4x\,\mathrm{e}^{\mathrm{i}qx}\\ &\quad\times\mathrm{Tr}\left[\varrho T_{\mathrm{c}}A^a_\mu(\varphi_i(x))A^b_\nu(\varphi_j(x))\right]/\mathrm{Tr}\varrho|_{\pi\text{ pole}}\,,\\ \varrho &= \mathrm{e}^{-\beta H},\end{aligned} \tag{30}$$

where T_c denotes time ordering with respect to the time contour of Fig. 8 in Appendix A. There we discuss at length how to obtain the matrix amplitude for an individual graph from its vacuum amplitude. To summarize, all we need is to replace the loop integrals $(J, J', J_{\mu\nu})$ encountered in the vacuum amplitudes by $(J^\beta, J'^\beta, J^\beta_{\mu\nu})$, where, in effect, the vacuum pion propagator is replaced by the 11- or 22-component of the thermal propagator. Further, the elements of the vacuum theory, namely, (Δ, Σ, Γ) need be replaced by the matrices $(\mathbf{\Delta}, \mathbf{\Sigma}, \mathbf{\Gamma})$ and also Δ^2 by $\mathbf{\Delta}\tau\mathbf{\Delta}$, where the matrices are given in Appendix A. Having obtained the matrix amplitude, we put in the factorized forms for all the matrices to get an equation among diagonal matrices, each of whose 11- and 22-elements are identical up to complex conjugation and possibly a $(-)$ sign. Thus we leave behind the matrix structure and work with the (single component) analytic amplitude $T^\beta_{\mu\nu}(q)$. Almost repeating (16) and (29) we write it as the sum of

$$T^{\beta(1+2+3)}_{\mu\nu} = q_\mu q_\nu F^2 \sum_{n=1}^{3} \left\{\gamma^\beta_n \mathrm{i}\Delta(q) + \sigma^\beta_n M^2 \Delta^2(q)\right\} + 8F^2\eta^2(l_1+2l_2)\left(q_\mu J^\beta_{\nu\lambda} + q_\nu J^\beta_{\mu\lambda}\right)q^\lambda \mathrm{i}\Delta(q)\,, \tag{31}$$

where γ^β_n and σ^β_n are obtained from γ_n and σ_n of (17), (18) and (20) after replacing the J by J^β, and

$$T^{\beta(4)}_{\mu\nu} = q_\mu q_\nu F^2\eta^2\left[\frac{2}{3}(J^\beta)^2 \mathrm{i}\Delta(q) - \frac{1}{6}\left\{8(J^\beta)^2 - K^\beta + 24 q^\varrho q^\sigma K^\beta_{\varrho\sigma}/M^2\right\}M^2\Delta^2(q)\right] + 4F^2\eta^2\left(q_\mu K^\beta_{\nu\lambda} + q_\nu K^\beta_{\mu\lambda}\right)q^\lambda \mathrm{i}\Delta(q)\,, \tag{32}$$

where, as in (27) and (28) for the vacuum case, we have expressed Γ^β and Σ^β in terms of $K^\beta, K^\beta_{\mu\nu}$ (and J^β) with

$$K^\beta(q) = \frac{\mathrm{i}}{M^2}\int \frac{\mathrm{d}^4k_1}{(2\pi)^4}\frac{\mathrm{d}^4k_2}{(2\pi)^4} \times \Delta_{11}(k_1)\Delta_{11}(k_2)\Delta_{11}(q-k_1-k_2) \tag{33}$$

and

$$K^\beta_{\mu\nu}(q) = \frac{\mathrm{i}}{M^4}\int \frac{\mathrm{d}^4k_1}{(2\pi)^4}\frac{\mathrm{d}^4k_2}{(2\pi)^4} k_{1\mu}k_{1\nu} \times \Delta_{11}(k_1)\Delta_{11}(k_2)\Delta_{11}(q-k_1-k_2)\,. \tag{34}$$

We must point out here that (33) and (34) hold only for the *real* parts. (The *imaginary* parts of both sides differ by the factor $(1+2n(|q_0|))^{-1}$, as follows from (A.27) and (A.31).) Since, however, we are interested only in the real parts of the pole parameters, our imprecise notation will lead to no error.

As already stated, the vacuum part of this thermal amplitude is of no interest to us here, beyond the one-loop results given in (14) and (15). In the β-dependent part, we have to isolate the finite terms from the divergent ones. To this end, we separate the $\mathbf{\Delta}_{11 \text{ or } 22}$ into its vacuum and thermal parts in the expressions for the J^β's and K^β's. In the case of J^β's we have simply

$$J^\beta = J + \bar J\,, \quad J'^\beta = J' + \bar J'\,, \quad J^\beta_{\mu\nu} = J_{\mu\nu} + \bar J_{\mu\nu}\,,$$

where

$$(\bar J, \bar J', \bar J_{\mu\nu}) = \int \frac{\mathrm{d}^4k}{(2\pi)^3} n(|k_0|)\left(\frac{1}{M^2}, -\frac{\partial}{\partial M^2}, \frac{k_\mu k_\nu}{M^4}\right) \times \delta(k^2 - M^2)\,. \tag{35}$$

K^β splits as follows:

$$K^\beta(q) = K(q) + K^\beta(q)\big|_{\mathrm{div}} + \bar K^J + \bar K(q)\,. \tag{36}$$

Here the second term gives the β-dependent divergent pieces, all proportional to λ. The finite, temperature dependent part can be expressed partly in terms of the $\bar J$'s defined above and the remainder as certain q-dependent integrals, constituting the third and the fourth term respectively. A similar decomposition holds for $K^\beta_{\mu\nu}$,

$$K^\beta_{\mu\nu}(q) = K_{\mu\nu}(q) + K^\beta_{\mu\nu}(q)\big|_{\mathrm{div}} + \bar K^J_{\mu\nu} + \bar K_{\mu\nu}(q)\,. \tag{37}$$

All the pieces in (36) and (37) are displayed in Appendix B. It is now easy to see that all the β-dependent divergent pieces cancel. We then get the complete, renormalized thermal amplitude:

$$T^\beta_{\mu\nu}(q) = T'_{\mu\nu}(q) + \bar T_{\mu\nu}(q)\,, \tag{38}$$

where $T'_{\mu\nu}$ is the vacuum amplitude without the free pole term, which is put in the β-dependent piece $\bar T_{\mu\nu}$,

$$\bar T_{\mu\nu}(q) = q_\mu q_\nu F^2\left[(1 - 2\bar J\eta + A\eta^2)\mathrm{i}\Delta(q) + \left\{\frac{1}{2}\bar J\eta - \left(B - \frac{\bar K(q)}{6} + \frac{q^\lambda q^\sigma}{M^2}S_{\lambda\sigma}(q)\right)\eta^2\right\}M^2\Delta^2(q)\right] + F^2\{q_\mu S_{\nu\lambda}(q) + q_\nu S_{\mu\lambda}(q)\}q^\lambda\eta^2\mathrm{i}\Delta(q)\,, \tag{39}$$

where the tensor $S_{\mu\nu}$ is given by

$$S_{\mu\nu}(q) = l\bar J_{\mu\nu} + 4\bar K_{\mu\nu}(q)\,, \tag{40}$$

and the (β-dependent) constants A and B are built out of $\bar J$ and $\bar J'$,

$$A = \bar J(2\bar J + l') + \bar J'\left(\bar J - \frac{\bar l_3}{16\pi^2}\right)\,, \quad B = \bar J\left(\frac{19}{8}\bar J + l''\right) + \frac{\bar J'}{4}\left(\bar J - \frac{\bar l_3}{16\pi^2}\right)\,. \tag{41}$$

The three combinations of coupling constants introduced above are

$$l = \frac{1}{12\pi^2}\left(\bar l_1 + 4\bar l_2 - \frac{14}{3}\right)\,, \quad l' = \frac{1}{48\pi^2}\left(6\bar l_1 + 4\bar l_2 - 9\bar l_4 - \frac{7}{3}\right)\,, \quad l'' = \frac{1}{48\pi^2}\left(6\bar l_1 + 4\bar l_2 - 6\bar l_3 - 3\bar l_4 - \frac{55}{12}\right)\,. \tag{42}$$

In (39) we have left out regular, non-pole pieces arising out of explicit factors of q^2. We remove further non-pole pieces by expanding the q-dependent functions in q_0 in the neighborhood of the pole $q_0^2 = \mathbf{q}^2 + M^2 \equiv \omega^2$ at fixed $\mathbf{q}$. Thus

$$\bar{K}(q) = \bar{K}^{(0)}(\omega) + \left(q_0^2 - \omega^2\right)\bar{K}^{(1)}(\omega) + \dots\,,$$
$$\bar{K}^{(1)}(\omega) = \frac{1}{2\omega}\frac{\partial}{\partial q_0}\bar{K}(q)\bigg|_{q_0=\omega}\,, \tag{43}$$

and similarly for $S_{\mu\nu}(q)$. The resulting expression may be put in the form

$$\bar{T}_{\mu\nu}(q) = -\frac{f_\mu(q) f_\nu(q)}{q_0^2 - \Omega^2(\omega)}\,, \tag{44}$$

where

$$\Omega^2(\omega) = \mathbf{q}^2 + M^2\bigg\{1 + \frac{1}{2}\bar{J}\eta - \left(B - \frac{\bar{K}^{(0)}}{6} + \frac{1}{M^2}(q^\lambda q^\sigma S_{\lambda\sigma})^{(0)} - \bar{J}^2\right)\eta^2\bigg\},$$
$$f_\mu(q) = F\bigg[q_\mu\bigg\{1 - \bar{J}\eta + \frac{1}{2}\left(A + \frac{M^2}{6}\bar{K}^{(1)} - (q^\lambda q^\sigma S_{\lambda\sigma})^{(1)} - \bar{J}^2\right)\eta^2\bigg\} + (S_{\mu\lambda}q^\lambda)^{(0)}\eta^2\bigg]\,. \tag{45}$$

Note that $f_\mu(q)$ is not proportional to q_μ due to the presence of the last term. This is due to the lack of Lorentz invariance in a medium, which serves as the preferred frame of reference. Thus, as in non-relativistic systems [13], we have here two different F, the temporal and the spatial ones [14],

$$f_0(q) = q_0 F^t(q)\,, \quad f_i(q) = q_i F^s(q)\,. \tag{46}$$

One may now find the thermal dispersion curve for the pion and its decay 'constants' at different values of $|\mathbf{q}|$. Instead, however, we set $\mathbf{q} = 0$ and find the effective mass and the decay constants as a function of temperature. Converting the parameters F and M to the physical values by (14) and (15) and using the result (B.9) of Appendix B, we finally get them as

$$M_\pi^2(T) = M_\pi^2\bigg\{1 + \frac{M_\pi^2}{2F_\pi^2}\bar{J} - \frac{M_\pi^4}{F_\pi^4}\left(l'''\bar{J} + \frac{11}{8}\bar{J}^2 + \frac{1}{4}\bar{J}\bar{J}' - \frac{\bar{K}}{6} + l\bar{J}_{00} + 4\bar{K}_{00}\right)\bigg\}\,, \tag{47}$$
$$F_\pi^t = F_\pi\bigg\{1 - \frac{M_\pi^2}{F_\pi^2}\bar{J} + \frac{M_\pi^4}{2F_\pi^4}\left(\bar{J}(\bar{J} + l') + \bar{J}\bar{J}' + \frac{M_\pi}{12}\frac{\partial\bar{K}}{\partial q_0} + l\bar{J}_{00} + 4\bar{K}_{00} - 2M_\pi\frac{\partial\bar{K}_{00}}{\partial q_0}\right)\bigg\}\,, \tag{48}$$

and

$$\left(F_\pi^t(T) - F_\pi^s(T)\right)/F_\pi = \frac{M_\pi^4}{3F_\pi^4}\big\{-12M_\pi C - l\bar{J} + 4\left(\bar{J}^2 - \bar{K} + l\bar{J}_{00} + 4\bar{K}_{00}\right)\big\}\,, \tag{49}$$

where l''' is given by

$$l''' = \frac{1}{192\pi^2}\left(24\bar{l}_1 + 16\bar{l}_2 - 27\bar{l}_3 - 24\bar{l}_4 - \frac{55}{3}\right)\,, \tag{50}$$

and C is the coefficient of the linear term in the expansion of $\bar{K}_{0i}(q)$ around $\mathbf{q} = 0$,

$$\bar{K}_{0i}(q) = C(q_0)q_i + \dots \tag{51}$$

Note that all the quantities $\bar{J}$, $\bar{K}$ etc. are now functions of M_π. These results agree with [11] except for the definition of l'''.[2]

5 Evaluation

We now need the values of the coupling constants, $\bar{l}_i$, $i = 1, \dots, 4$. They were already determined in the original work [4, 5], but all of them are not accurate enough [15]. The best values obtained so far follow from matching the dispersion theoretic phenomenological representation for the $\pi\pi$ scattering amplitude to its two-loop evaluation in chiral perturbation theory [16],

$$\bar{l}_1 = -0.4 \pm 0.6\,, \quad \bar{l}_2 = 4.3 \pm 0.1\,, \qquad \text{(two loop)}$$

while the values relevant in the context of one-loop approximation are

$$\bar{l}_1 = -1.9 \pm 0.2\,, \quad \bar{l}_2 = 5.25 \pm 0.04\,. \qquad \text{(one loop)}$$

The difference in the two sets of values are attributed to the infrared singularities that can be better dealt with in the two-loop matching than in the case of one loop. The original crude estimate of $\bar{l}_3$ [4, 5],

$$\bar{l}_3 = 2.9 \pm 2.4 \qquad \text{(one loop)},$$

has not been improved further. Finally the two-loop estimate of $\bar{l}_4$ [16],

$$\bar{l}_4 = 4.4 \pm 0.2 \qquad \text{(two loop)},$$

does not differ much from the original one-loop estimate [4, 5],

$$\bar{l}_4 = 4.3 \pm 0.9 \qquad \text{(one loop)},$$

[2] In the expression for $192\pi^2 l'''$, Toublan [11] finds the coefficients of $\bar{l}_3$ and $\bar{l}_4$ to be -15 and -12 respectively, instead of -27 and -24 found by us

as the presence of infrared singularities is weakly felt here.

It should be noted here that in our two-loop calculation of the pion pole in the axial-vector Green's function at finite temperature, it is actually the scattering amplitude in vacuum to one loop that enters its temperature dependent part. It is thus appropriate to use the one-loop estimate of the coupling constants in the present context.

We now evaluate the pion pole parameters in two regions of temperature. First consider the so-called high temperature limit, $T \gg M_\pi$. To remain within the domain of the low temperature expansion, this limit is implemented not by letting T increase, but instead by holding T fixed and sending M_π to zero. The value of M_π is determined by the quark masses. Thus the high temperature limit is equivalent to the chiral limit of QCD theory.

The values of the relevant integrals in the chiral limit are given in Appendix C. The contributing terms are only $\eta^2 \bar{J}^2$ and the combination

$$\eta^2(l\bar{J}_{00} + 4\bar{K}_{00}) = \frac{T^4}{36F^4}(Z(T)+c)\,, \tag{52}$$

where

$$Z(T) = \ln\frac{M}{T} + \frac{1}{10}(\bar{l}_1 + 4\bar{l}_2)\,,$$
$$c = -\frac{7}{15} - \ln 2 + 1 - I_1 + I_2 = 0.30\,,$$

as obtained from (42) and (C.1). Here $Z(T)$ has actually no logarithmic singularity in the chiral limit, as can be seen by shifting the renormalization scale of the coupling constants with (6) from M to any other value μ.

We thus get the results for the pion mass and decay constants to two loops at finite temperature in the chiral limit as follows:

$$\left.\frac{M_\pi^2(T)}{M_\pi^2}\right|_\chi = 1 + \frac{T^2}{24F^2} - \frac{T^4}{36F^4}\left(Z(T) + c + \frac{11}{32}\right)$$
$$= 1 + \frac{T^2}{24F^2} - \frac{T^4}{36F^4}\ln\frac{\Lambda_M}{T}\,, \tag{53}$$
$$\left.\frac{F_\pi^t(T)}{F_\pi}\right|_\chi = 1 - \frac{T^2}{12F^2} + \frac{T^4}{72F^4}\left(Z(T) + c + \frac{1}{4}\right)$$
$$= 1 - \frac{T^2}{12F^2} + \frac{T^4}{72F^4}\ln\frac{\Lambda_F}{T}\,, \tag{54}$$
$$\left.\frac{F_\pi^t(T) - F_\pi^s(T)}{F_\pi}\right|_\chi = \frac{T^4}{27F^4}\left(Z(T) + c + \frac{1}{4}\right)$$
$$= \frac{T^4}{27F^4}\ln\frac{\Lambda_\Delta}{T}\,, \tag{55}$$

where the logarithmic scales are

$$\Lambda_M = 1.8\,\text{GeV}\,, \quad \Lambda_F = \Lambda_\Delta = 1.6\,\text{GeV}\,,$$

in agreement with [11]. Note that Λ_F is associated with $F_\pi^t(T)$ and not its square, as in this reference. (Had we chosen the two-loop coupling constants, we would get somewhat smaller values for the Λ, namely, $\Lambda_M = 1.4\,\text{GeV}$, $\Lambda_F = \Lambda_\Delta = 1.3\,\text{GeV}$.)

Next we consider the low temperature limit, $T \ll M_\pi$. We shall express all quantities in terms of the dimensionless ratio

$$\tau = \frac{T}{M_\pi}$$

times possibly a power of temperature.

The leading behavior of all the pole parameters in the low temperature region is given essentially by that of $\bar{J}$, as seen from (C.3)–(C.5). Thus we get

$$\left.\frac{M_\pi^2(T)}{M_\pi^2}\right|_\tau = 1 + \left(\frac{M_\pi^2}{2F_\pi^2} - \frac{5g_1 M_\pi^4}{24\pi^2 F_\pi^4}\right)\bar{J}|_\tau\,, \tag{56}$$
$$\left.\frac{F_\pi^t(T)}{F_\pi}\right|_\tau = 1 - \left(\frac{M_\pi^2}{F_\pi^2} - \frac{5g_2 M_\pi^4}{48\pi^2 F_\pi^4}\right)\bar{J}|_\tau\,, \tag{57}$$
$$\left.\frac{F_\pi^t(T) - F_\pi^s(T)}{F_\pi}\right|_\tau = \left.\frac{g_3 M_\pi^4}{12\pi^2 F_\pi^4}\bar{J}\right|_\tau\,, \tag{58}$$

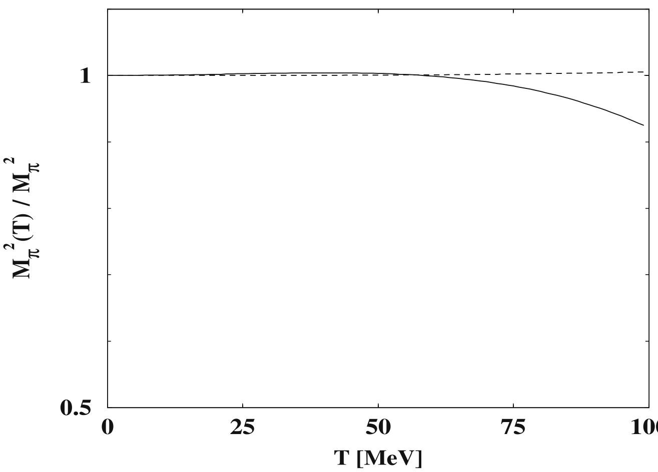

Fig. 5. Thermal pion mass squared in chiral (*continuous curve*) and non-relativistic (*dashed one*) limits

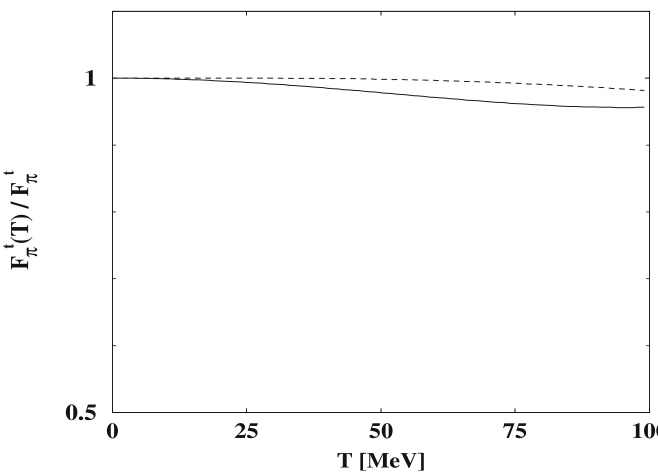

Fig. 6. 'Temporal' type of thermal pion decay constant in the two limits as in Fig. 5

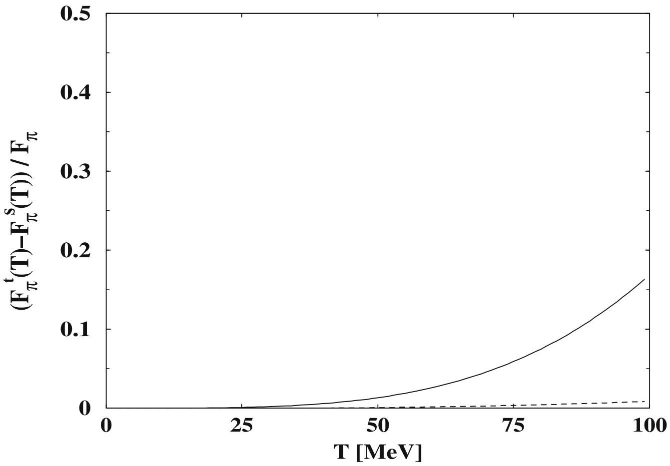

Fig. 7. Difference of temporal and spatial types of pion decay constant in the two limits as in Figs. 5 and 6

where

$$\begin{aligned} g_1 &= \bar{l}_1 + 2\bar{l}_2 - \frac{27}{40}\bar{l}_3 - \frac{3}{5}\bar{l}_4 + \frac{9}{8}\,, \\ g_2 &= \bar{l}_1 + 2\bar{l}_2 - \frac{9}{10}\bar{l}_4 + \frac{3}{10}\,, \\ g_3 &= \bar{l}_1 + 4\bar{l}_2 + \frac{4}{3}\,, \end{aligned} \tag{59}$$

and $\bar{J}|_\tau$ is given by (C.3).

It is interesting to note that the two-loop contributions to the effective parameters are always of opposite sign compared to that of one loop, when it contributes. Figures 5–7 show the temperature dependence of these parameters.

6 Discussion

Though the propagators, vertices and self-energies assume the form of 2×2 matrices in the real time field theory at finite temperature, each of them is essentially given by a single analytic function, as can be seen from an appropriate factorization of these matrices. The same is true of the ensemble average of (the T-product of) any two operators. Here we show that one can find its thermal 2×2 matrix amplitude directly from the vacuum amplitude and take advantage of this factorization to get the analytic (single component) thermal amplitude. Thus compared to the commonly followed practice of considering the 11-element of the thermal matrix, the use of matrices not only simplifies and frees the calculation from ill-defined quantities at intermediate steps, but also yields directly the amplitude with proper analytic properties, not possessed by the 11-element.

In this work we use this matrix method to calculate the thermal pion pole term in the axial-vector two-point function in the framework of chiral perturbation theory. From the analytic amplitude we derive the effective mass and the decay constants of the pion at finite temperature. These are evaluated in two limits, the chiral and the non-relativistic one. The two evaluations agree rather closely up to about $T \simeq 100\,\mathrm{MeV}$.

The dynamical degrees of freedom in the effective Lagrangian are only those of the Goldstone bosons. Thus one may think, a priori, that the massive states also contribute, as virtual states in Feynman diagrams and as real particles in the heat bath. The contribution of such virtual states are, however, already incorporated in the low energy constants, determined phenomenologically [4, 5, 16]. In fact, the calculation of these contributions show that they almost saturate the values of these constants [4, 5, 17, 18].

On the other hand, the massive states in the heat bath do contribute, though exponentially (like $\mathrm{e}^{-m_\varrho/T}$ for the ϱ meson, for example); they do not show up at any finite order in a power series in T. However, because of the exponential suppression and the presence of a factor of T^2 from interaction, such terms can contribute at most 5% in different physical quantities for T less than 100 MeV [12].

In this work we are concerned only with the real parts of the effective parameters. As expected, their imaginary parts also lead to interesting physical quantities. The imaginary part of the pion pole position gives directly the mean free path in thermalized matter. It has been evaluated by using chiral perturbation theory [7] and by using the virial formula [19]. The imaginary parts of $F_\pi^t(T)$ and $F_\pi^s(T)$ are related through this mean free path [14]. We shall report on their evaluation elsewhere.

Finally we compare our work with that of Toublan [11], whom we follow at a number of points. He obtains the thermal amplitude in a somewhat intuitive manner, while we formulate rules to write the matrix amplitude, which leads immediately to the analytic thermal amplitude. These rules, in effect, justify his way of writing the thermal amplitude, as far as its real part is concerned. Our results (47)–(49) for the effective mass and decay constants of the pion agree with his, except for the coefficient l''' in the equation for $M_\pi^2(T)$ – (see footnote 2). But their chiral limits (53)–(55) agree completely with his, as the term with this coefficient does not contribute in this limit. We also find the effective parameters in the non-relativistic limit, in which this term does contribute.

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Appendix A

The feature of the real time thermal field theory that distinguishes it from the vacuum theory is the time contour in their generating functionals. While it is the infinite real line for the vacuum theory, it must be augmented with a return path for the thermal theory. One example of such a path that we shall use is shown in Fig. 8. The return path may be folded onto the onward path, generating fields with a displaced time argument. Thus compared with the vacuum

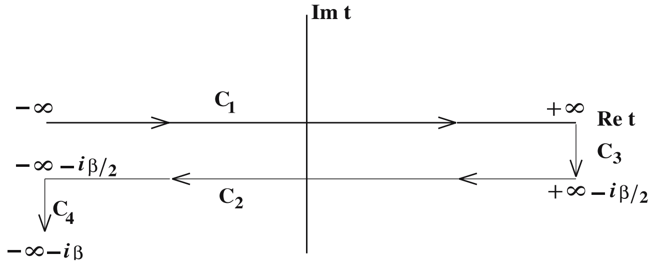

Fig. 8. The complex time contour of real time thermal field theory

generating functional,

$$\left\langle 0\middle| T\exp\left[\mathrm{i}\int \mathrm{d}^3x\int_{-\infty}^{\infty}\mathrm{d}t \times\left\{\mathcal{L}_{\mathrm{int}}(\varphi)+j(x)\varphi(x)+j^{\mu}(x)A_{\mu}(x)\right\}_{\mathrm{in}}\right]\middle|0\right\rangle, \quad (A.1)$$

the thermal one is given by

$$\mathrm{Tr}\left[\varrho T\exp\left[\mathrm{i}\int \mathrm{d}^3x\int_{-\infty}^{\infty}\mathrm{d}t\Big\{\mathcal{L}_{\mathrm{int}}(\varphi_1)-\mathcal{L}_{\mathrm{int}}(\varphi_2) + \sum_{n=1,2}(j_n(x)\varphi_n(x)+j_n^{\mu}(x)A_{\mu n}(x))\Big\}_{\mathrm{in}}\right]\right], \quad (A.2)$$

both written in the interaction representation in terms of the in-fields, denoted by the subscript 'in', which we shall omit below. The fields $\varphi_1(x)$ and $\varphi_2(x)$ have their time arguments on the segments C_1 and C_2 respectively: $\varphi_1(x)=\varphi(\mathbf{x},t)$ is the 'physical field' and $\varphi_2(x)=\varphi(\mathbf{x},t-\mathrm{i}\beta/2)$ is the 'ghost' field. The $(-)$ sign before $\mathcal{L}(\varphi_2)$ is forced upon us by the theory, but the signs before the $n=2$ terms with the external fields are at our disposal, which we choose to be positive. The pieces C_3 and C_4 of the complex time contour are of no consequence and have been dropped.

The two sets of fields make any thermal two-point function a 2×2 matrix. In particular, the free pion propagator is

$$\mathbf{\Delta}^{ab}(x-y)=\delta^{ab}\mathbf{\Delta}(x-y) = \begin{pmatrix}\langle T\varphi_1^a(x)\varphi_1^b(y)\rangle & \langle\varphi_2^b(y)\varphi_1^a(x)\rangle\\ \langle\varphi_2^a(x)\varphi_1^b(y)\rangle & \langle\widetilde{T}\varphi_2^a(x)\varphi_2^b(y)\rangle\end{pmatrix}, \quad (A.3)$$

where $\widetilde{T}$ denotes anti-time ordering, can be evaluated directly in momentum space as

$$\mathbf{\Delta}(q)= \begin{pmatrix}\Delta(q)+2\pi n(|q_0|)\delta(q^2-M^2) & 2\pi n(|q_0|)\delta(q^2-M^2)\mathrm{e}^{\beta|q_0|/2}\\ 2\pi n(|q_0|)\delta(q^2-M^2)\mathrm{e}^{\beta|q_0|/2} & \Delta^*(q)+2\pi n(|q_0|)\delta(q^2-M^2)\end{pmatrix}, \quad (A.4)$$

where $n(|q_0|)=(\mathrm{e}^{\beta|q_0|}-1)^{-1}$ is the pion distribution function and $\Delta(q)=\mathrm{i}/(q^2-M^2+\mathrm{i}\varepsilon)$ is the pion propagator in vacuum. (A boldface letter will always indicate a 2×2 matrix.)

Given the Feynman amplitude for any two-point function in vacuum, it is simple to write the corresponding thermal matrix amplitude. The correspondence may be found by comparing the Wick contractions for graphs in the two cases, with particular attention to the $(-)$ sign before the 'ghost' Lagrangian. As an example, consider the graph (d) of Fig. 1, for which we show this correspondence in detail. Its vacuum amplitude in coordinate space is obtained from

$$-F^2\mathrm{i}\int \mathrm{d}^4z\langle 0|T\partial_\mu\varphi^a(x)\partial_\nu\varphi^b(y)\mathcal{L}_{\mathrm{int}}(\varphi(z))|0\rangle\,, \quad (A.5)$$

where $\mathcal{L}_{\mathrm{int}}$ is a piece in $\mathcal{L}^{(2)}$,

$$\mathcal{L}_{\mathrm{int}}(\varphi)=-\frac{1}{6F^2}\Big\{\vec\varphi\cdot\vec\varphi\partial_\mu\vec\varphi\cdot\partial^\mu\vec\varphi-\vec\varphi\cdot\partial_\mu\vec\varphi\vec\varphi\cdot\partial^\mu\vec\varphi -\frac{M^2}{4}(\vec\varphi\cdot\vec\varphi)^2\Big\}, \quad (A.6)$$

the field $\vec\varphi$ denoting the pion iso-vector triplet $(\varphi^1,\varphi^2,\varphi^3)$. It gives the amplitude in momentum space,

$$T_{\mu\nu}^{(1d)}(q)=q_\mu q_\nu J(M)M^2\left\{\frac{2}{3}\mathrm{i}\Delta(q)+\frac{1}{2}M^2\Delta^2(q)\right\}, \quad (A.7)$$

where $J(M)$ is defined by (9). The single propagator in this expression arises from the cancellation,

$$(q^2-M^2)\Delta^2(q)=\mathrm{i}\Delta(q)\,. \quad (A.8)$$

We now identify the contractions in (A.5) that produce this result. To focus on the contractions, we omit the derivatives and isospin indices on the pion fields and write schematically a term of the matrix element (A.5) as

$$\langle 0|T\varphi(x)\varphi(y)\varphi^4(z)|0\rangle\sim J(M)\Delta(x-z)\Delta(z-y) \sim J(M)\Delta^2(q) \quad (A.9)$$

in momentum space. To get the corresponding thermal matrix amplitude, consider its ij element,

$$-F^2\mathrm{i}\int \mathrm{d}^4z\langle T\partial_\mu\varphi_i^a(x)\partial_\nu\varphi_j^b(y) \times\{\mathcal{L}_{\mathrm{int}}(\varphi_1(z))-\mathcal{L}_{\mathrm{int}}(\varphi_2(z))\}\rangle\,. \quad (A.10)$$

Again we write schematically a term of this matrix element and contract its fields as follows:

$$\begin{aligned}&\langle T\varphi_i(x)\varphi_j(y)\{\varphi_1^4(z)-\varphi_2^4(z)\}\rangle\\ &\sim J^\beta(M)\langle T\varphi_i(x)\varphi_j(y)\{\varphi_1^2(z)-\varphi_2^2(z)\}\rangle\\ &\sim J^\beta(M)\{\mathbf{\Delta}_{i1}(x-z)\mathbf{\Delta}_{1j}(z-y)-\mathbf{\Delta}_{i2}(x-z)\mathbf{\Delta}_{2j}(z-y)\}\\ &\sim J^\beta(M)(\mathbf{\Delta}(q)\tau\mathbf{\Delta}(q))_{ij} \quad (A.11)\end{aligned}$$

in momentum space, where we use in the second line the fact that the contractions of two φ_1 and two φ_2 at the same point yield the same result,

$$J^\beta(M)=\frac{1}{M^2}\int\frac{\mathrm{d}^4k}{(2\pi)^4}\mathbf{\Delta}(q)_{11\text{ or }22} \quad (A.12)$$

and the matrix τ is

$$\tau = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} . \quad \text{(A.13)}$$

Note that a cancellation similar to (A.8) for the vacuum case works also here,

$$(q^2 - M^2)\mathbf{\Delta}(q)\tau\mathbf{\Delta}(q) = \mathrm{i}\mathbf{\Delta}(q) . \quad \text{(A.14)}$$

Comparing the contractions (A.9) and (A.11), we see that the thermal matrix amplitude of graph (1d) can be obtained from the vacuum amplitude (A.7) simply by replacing J with J^β, Δ with $\mathbf{\Delta}$ and Δ^2 by $\mathbf{\Delta}\tau\mathbf{\Delta}$ in it.

A little reflection on the other graphs of Figs. 1–3 will convince us that the thermal amplitudes of all these graphs may be obtained from their vacuum amplitudes by the replacements just stated above, together with J' and $J_{\mu\nu}$ by J'^β and $J^\beta_{\mu\nu}$ respectively, where

$$J'^\beta(M) = \mathrm{i}\int \frac{\mathrm{d}^4k}{(2\pi)^4}(\mathbf{\Delta}(k)\tau\mathbf{\Delta}(k))_{11 \text{ or } 22} = -\frac{\partial}{\partial M^2}(M^2 J^\beta) , \quad \text{(A.15)}$$

and

$$J^\beta_{\mu\nu}(M) = \frac{1}{M^4}\int \frac{\mathrm{d}^4k}{(2\pi)^4} k_\mu k_\nu \mathbf{\Delta}(k)_{11 \text{ or } 22} . \quad \text{(A.16)}$$

In (A.15) we use the so-called mass-derivative formula for the matrix propagator [20],

$$(\mathbf{\Delta}\tau)^2 = \mathrm{i}\frac{\partial}{\partial M^2}(\mathbf{\Delta}\tau) , \quad \text{(A.17)}$$

which is the thermal extension of the trivial relation $\Delta^2(q) = \mathrm{i}\partial\Delta/\partial M^2$ for the vacuum propagator.

So long we constructed thermal amplitudes with the matrix propagator only, inserting τ explicitly to account for the $(-)$ sign from the 'ghost' Lagrangian. We now introduce two kinds of parts of graphs, namely the self-energy and the (two-point) vertices, where it is convenient to include the effect of the associated $(-)$ sign(s) in their definitions. Thus writing the ij-component of the matrix amplitude of Fig. 4b, again schematically, and contracting the fields, we get

$$\left\langle T\varphi_i(x)\left\{\varphi_1^4(u) - \varphi_2^4(u)\right\}\left\{\varphi_1^4(v) - \varphi_2^4(v)\right\}\varphi_j(y)\right\rangle \sim (\mathbf{\Delta}(x-u)\mathbf{\Sigma}(u-v)\mathbf{\Delta}(v-y))_{ij} , \quad \text{(A.18)}$$

and absorbing the $(-)$ signs in the definition of $\mathbf{\Sigma}$,

$$\mathbf{\Sigma} = \begin{pmatrix} s_{11} & -s_{12} \\ -s_{21} & s_{22} \end{pmatrix} , \quad s_{ij} = (\mathbf{\Delta}_{ij})^3. \quad \text{(A.19)}$$

Likewise, for the graph of Fig. 4a we write

$$\left\langle T\varphi_i^3(x)\left\{\varphi_1^4(u) - \varphi_2^4(u)\right\}\varphi_j(y)\right\rangle \sim (\mathbf{\Gamma}^{(1)}\mathbf{\Delta})_{ij} , \quad \mathbf{\Gamma}^{(1)} = \begin{pmatrix} s_{11} & -s_{12} \\ s_{21} & -s_{22} \end{pmatrix} , \quad \text{(A.20)}$$

and

$$\left\langle T\varphi_i(x)\left\{\varphi_1^4(u) - \varphi_2^4(u)\right\}\varphi_j^3(y)\right\rangle \sim (\mathbf{\Delta}\mathbf{\Gamma}^{(2)})_{ij} , \quad \mathbf{\Gamma}^{(2)} = \begin{pmatrix} s_{11} & s_{12} \\ -s_{21} & -s_{22} \end{pmatrix} . \quad \text{(A.21)}$$

The matrix amplitudes are greatly simplified by factoring out matrices involving only the pion distribution function. Thus the free propagator given by (A.4) can be factored as follows:

$$\mathbf{\Delta}(q) = \mathbf{U}(q)\begin{pmatrix} \Delta(q) & 0 \\ 0 & \Delta^*(q) \end{pmatrix}\mathbf{U}(q) , \quad \mathbf{U}(q) = \begin{pmatrix} \sqrt{1+n} & \sqrt{n} \\ \sqrt{n} & \sqrt{1+n} \end{pmatrix} . \quad \text{(A.22)}$$

Also the full propagator $\mathbf{\Delta}'$ and the two-point function $\mathbf{T}_{\mu\nu}$ admit similar factorizations,

$$\mathbf{\Delta}'(q) = \mathbf{U}(q)\begin{pmatrix} \Delta'^\beta(q) & 0 \\ 0 & \Delta'^{\beta *}(q) \end{pmatrix}\mathbf{U}(q) , \quad \mathbf{T}_{\mu\nu}(q) = \mathbf{U}(q)\begin{pmatrix} T^\beta_{\mu\nu}(q) & 0 \\ 0 & -T^{\beta *}_{\mu\nu}(q) \end{pmatrix}\mathbf{U}(q) , \quad \text{(A.23)}$$

as is suggested by the evaluation of our graphs. More rigorously, these follow from their spectral representations.

To derive a similar factorization of the self-energy part $\mathbf{\Sigma}$, we look at the Dyson–Schwinger equation for the full propagator,

$$\mathbf{\Delta}' = \mathbf{\Delta} + \mathbf{\Delta}(-\mathrm{i}\mathbf{\Sigma})\mathbf{\Delta}' . \quad \text{(A.24)}$$

Inserting the factorizations for $\mathbf{\Delta}'$ and $\mathbf{\Delta}$, we infer that $\mathbf{\Sigma}$ must have the factorized form [21],

$$\mathbf{\Sigma}(q) = \mathbf{U}^{-1}\begin{pmatrix} \Sigma^\beta(q) & 0 \\ 0 & -\Sigma^{\beta *}(q) \end{pmatrix}\mathbf{U}^{-1} . \quad \text{(A.25)}$$

It immediately follows that

$$\mathbf{\Sigma}_{22} = -\mathbf{\Sigma}_{11}^* , \quad \mathbf{\Sigma}_{21} = \mathbf{\Sigma}_{12} . \quad \text{(A.26)}$$

Further we can get the function Σ^β entirely from $\mathbf{\Sigma}_{11}$,

$$\mathrm{Re}\,\Sigma^\beta = \mathrm{Re}\,\mathbf{\Sigma}_{11} , \quad \mathrm{Im}\,\Sigma^\beta = \frac{1}{1+2n}\mathrm{Im}\,\mathbf{\Sigma}_{11} . \quad \text{(A.27)}$$

In the same way the relations

$$\mathbf{T} \sim -\mathrm{i}\mathbf{\Gamma}^{(1)}\mathbf{\Delta} \sim \mathbf{\Delta}(-\mathrm{i}\mathbf{\Gamma}^{(2)}) \quad \text{(A.28)}$$

give us the factorizations

$$\mathbf{\Gamma}^{(1)} = \mathbf{U}\begin{pmatrix} \Gamma^\beta & 0 \\ 0 & \Gamma^{\beta *} \end{pmatrix}\mathbf{U}^{-1} , \quad \mathbf{\Gamma}^{(2)} = \mathbf{U}^{-1}\begin{pmatrix} \Gamma^\beta & 0 \\ 0 & \Gamma^{\beta *} \end{pmatrix}\mathbf{U}. \quad \text{(A.29)}$$

We see that $\mathbf{\Gamma}^{(1)}$ and $\mathbf{\Gamma}^{(2)}$ differ by a $(-)$ sign in the off-diagonal elements, which is of no consequence to us. We thus omit the superscripts to write the relations given by (A.29) as

$$\mathbf{\Gamma}_{22} = \mathbf{\Gamma}_{11}^{*}, \quad \mathbf{\Gamma}_{21} = \mathbf{\Gamma}_{12}. \tag{A.30}$$

Again the 11-element determines the function Γ^{β} completely,

$$\mathrm{Re}\,\Gamma^{\beta} = \mathrm{Re}\,\mathbf{\Gamma}_{11}, \quad \mathrm{Im}\,\Gamma^{\beta} = \frac{1}{1+2n}\,\mathrm{Im}\,\mathbf{\Gamma}_{11}. \tag{A.31}$$

Appendix B

The β-dependent, divergent and finite parts of $K^{\beta}(q)$ and $K^{\beta}_{\mu\nu}$ have been obtained in [11], which we reproduce here for completeness. The divergent parts reside only in terms linear in the distribution function, where we need the vacuum integrals,

$$\begin{aligned} L(p) &= \mathrm{i}\int \frac{\mathrm{d}^4k}{(2\pi)^4}\Delta(k)\Delta(p-k) \\ &= -2\lambda - \frac{1}{16\pi^2} + R(p), \end{aligned}$$

and

$$\begin{aligned} L_{\mu\nu}(p) &= \frac{\mathrm{i}}{M^2}\int \frac{\mathrm{d}^4k}{(2\pi)^4}k_\mu k_\nu \Delta(k)\Delta(p-k) \\ &= \frac{\lambda}{6M^2}\{(p^2-6M^2)g_{\mu\nu} - 4p_\mu p_\nu\} \\ &\quad - \frac{1}{2(24\pi M)^2}\big[20p_\mu p_\nu - 2p^2 g_{\mu\nu} \\ &\quad + \{4p_\mu p_\nu - (p^2-4M^2)g_{\mu\nu} - 4M^2 p_\mu p_\nu/p^2\}R(p)\big], \end{aligned} \tag{B.1}$$

with

$$R(p) = -\frac{1}{16\pi^2}\int_0^1 \mathrm{d}x\,\ln(1-p^2x(1-x)/M^2). \tag{B.2}$$

Then the terms linear in n are given by

$$\begin{aligned} K^{\beta}(q)|_n &= \frac{3}{M^2}\int \frac{\mathrm{d}^4k}{(2\pi)^3}\delta(k^2-M^2)n(|k_0|)L(q-k), \\ &= -6\lambda\bar{J} - \frac{3}{16\pi^2}\bar{J} \\ &\quad + \frac{3}{M^2}\int \frac{\mathrm{d}^4k}{(2\pi)^3}\delta(k^2-M^2)n(|k_0|)R(q-k) \end{aligned} \tag{B.3}$$

and

$$\begin{aligned} K^{\beta}_{\mu\nu}(q)|_n &= \frac{1}{M^4}\int \frac{\mathrm{d}^4k}{(2\pi)^3}k_\mu k_\nu \delta(k^2-M^2)n(|k_0|)L(q-k) \\ &\quad + \frac{2}{M^2}\int \frac{\mathrm{d}^4k}{(2\pi)^3}k_\mu k_\nu \delta(k^2-M^2)n(|k_0|)L_{\mu\nu}(q-k) \\ &= -\frac{\lambda}{3}\big[10\bar{J}_{\mu\nu} + \{5g_{\mu\nu} + (4q_\mu q_\nu - q^2 g_{\mu\nu})/M^2\}\bar{J}\big] \\ &\quad + \frac{1}{2(12\pi M)^2}\big[\{(q^2+M^2)g_{\mu\nu} - 10q_\mu q_\nu\}\bar{J} - 28M^2\bar{J}_{\mu\nu}\big] \\ &\quad + \frac{1}{6M^4}\int \frac{\mathrm{d}^4k}{(2\pi)^3}\delta(k^2-M^2)n(|k_0|)R(q-k) \\ &\quad \times \big[4q_\mu q_\nu - 4(q_\mu k_\nu + q_\nu k_\mu) + 10k_\mu k_\nu \\ &\quad - g_{\mu\nu}(q^2 - 3M^2 - 2qk) \\ &\quad - 4M^2(q-k)_\mu (q-k)_\nu/(q-k)^2\big]. \end{aligned} \tag{B.4}$$

Next consider the terms quadratic in n. Introducing the integrals

$$\begin{aligned} Q(p) &= \int \frac{\mathrm{d}^4k}{(2\pi)^3}\frac{\delta(k^2-M^2)n(k)}{(p-k)^2-M^2}, \\ Q_\mu(p) &= \int \frac{\mathrm{d}^4k}{(2\pi)^3}k_\mu \frac{\delta(k^2-M^2)n(k)}{(p-k)^2-M^2}, \end{aligned} \tag{B.5}$$

we can write them as

$$K^{\beta}(q)|_{n^2} = -\frac{3}{M^2}\int \frac{d^4k}{(2\pi)^3}\delta(k^2-M^2)n(|k_0|)Q(q-k) \tag{B.6}$$

and

$$\begin{aligned} K^{\beta}_{\mu\nu}(q)|_{n^2} &= -\frac{1}{M^4}\int \frac{\mathrm{d}^4k}{(2\pi)^3}\delta(k^2-M^2)n(|k_0|) \\ &\quad \times [\{q_\mu q_\nu + 4k_\mu k_\nu - 2(q_\mu k_\nu + q_\nu k_\mu)\}Q(q-k) \\ &\quad + k_\mu Q_\nu(q-k) + k_\nu Q_\mu(q-k)]. \end{aligned} \tag{B.7}$$

These terms as well as the last terms in (B.3) and (B.4) belong to $\bar{K}(q)$ and $\bar{K}_{\mu\nu}(q)$ in the notation of (36) and (37).

Note that Q_i and Q_0 are not independent but are related by

$$Q_i(p) = \frac{p_i}{2|\mathbf{p}|^2}\{2p_0 Q_0(p) - p^2 Q(p) + \bar{J}\}. \tag{B.8}$$

Using this relation one can relate $\bar{K}_{ij}$ and $\bar{K}_{00}$ at the pole, $q_0 = M$ with $\mathbf{q} = 0$,

$$\bar{K}_{ij} = \frac{\delta_{ij}}{3}\left(\bar{K}_{00} - \bar{K} + \bar{J}^2\right). \tag{B.9}$$

Being interested in the real parts, we shall not consider terms cubic in n, which are imaginary.

Appendix C

Here we write the chiral and the non-relativistic limits of the integrals occurring in the expressions for the effective pion parameters. The chiral limits were obtained in [11][3]:

$$\begin{aligned} \eta\bar{J}\big|_\chi &= \frac{T^2}{12F^2} - \frac{MT}{4\pi F^2}, \quad \eta^2\bar{J}_{00}\big|_\chi = \frac{\pi^2T^4}{30F^4}, \\ \eta^2\bar{K}_{00}\big|_\chi &= \frac{T^4}{144F^4}\left(\ln\frac{M}{T} - \ln 2 + 1 - I_1 + I_2\right), \end{aligned} \tag{C.1}$$

[3] In [11] the integral I_1 is evaluated by expressing it in terms of derivatives of Zeta and Gamma functions. But since the integral I_2 has to be evaluated numerically anyway, we can do the same for I_1 also and get an identical result.

where

$$I_1 = \frac{15}{2\pi^4}\int_0^\infty \frac{\mathrm{d}x x^3 \ln x}{\mathrm{e}^x - 1} = 0.60\,,$$

$$I_2 = \frac{18}{\pi^4}\int_0^\infty \frac{\mathrm{d}x x^3}{\mathrm{e}^x - 1}\int_0^1 \frac{\mathrm{d}\alpha}{\mathrm{e}^{\alpha x} - 1} \times \left\{(1+\alpha^2)\ln\frac{1+\alpha}{1-\alpha} + \alpha \ln \frac{1-\alpha^2}{\alpha^2 x^2}\right\} = 1.05\,. \quad \text{(C.2)}$$

The integrals for $\eta^2 \bar{K}\big|_\chi$ and $\eta^2 \frac{\partial \bar{K}}{\partial q_0}\Big|_\chi$ vanish, while those for $\eta^2 \frac{\partial \bar{K}_{00}}{\partial q_0}\Big|_\chi$ and $\eta^2 C\big|_\chi$ are finite in the chiral limit. (Actually the terms linear and quadratic in n in each of the latter two quantities have singular pieces separately in this limit, but they cancel out in their respective sums.)

Next we calculate the integrals in the low temperature region, where $\tau \equiv T/M_\pi \ll 1$. Keeping only the leading terms, we get

$$\bar{J}\big|_\tau = \bar{J}_{00}\big|_\tau = \left(\frac{\tau}{2\pi}\right)^{3/2} \mathrm{e}^{-\frac{1}{\tau}}\,, \quad \text{(C.3)}$$

$$\frac{1}{3}\bar{K}\big|_\tau = \frac{1}{3}\bar{K}_{00}\big|_\tau = \frac{1}{2}M_\pi \frac{\partial \bar{K}_{00}}{\partial q_0}\Big|_\tau = M_\pi C\big|_\tau = \frac{1}{16\pi^2}\bar{J}\big|_\tau\,, \quad \text{(C.4)}$$

$$M_\pi \frac{\partial \bar{K}}{\partial q_0}\Big|_\tau = O(\tau^{5/2}\mathrm{e}^{-\frac{1}{\tau}})\,. \quad \text{(C.5)}$$

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